Ethiopian International Journal of Multidisciplinary Research

pISSN:2349-5707 Volume:12,Issue 12,December-2025 eISSN:2349-5715

WAITING TIME DISTRIBUTIONS FOR THE DOUBLING MAP

Abdusalomov Khasanboy

National University of Uzbekistan,PhD

e-mail: hasanboymat2016@gmail.com , +99890-038-03-20

Introduction

The study of waiting times, also referred to as hitting times or return times, plays a central role in ergodic theory, probability theory, and the theory of dynamical systems. Given a measurable dynamical system and a target set of small measure, one is naturally led to ask how long a typical orbit needs to wait before entering this set for the first time. This problem has deep connections with recurrence theory, limit laws, symbolic dynamics, and extreme value theory.

A classical motivation originates from the Poincaré recurrence theorem, which guarantees that almost every point returns infinitely often to any set of positive measure. However, Poincaré recurrence is qualitative in nature and provides no information about the distribution or growth rate of return times. Quantitative refinements of recurrence phenomena were initiated by Kac, who established his celebrated formula relating the expected return time to the inverse of the measure of the set.

In recent decades, substantial progress has been made in understanding the statistical behavior of waiting times for strongly mixing dynamical systems. For a wide class of systems, including expanding maps and shifts of finite type, it has been shown that suitably normalized waiting times converge in distribution to an exponential law. Such results were obtained by Galves and Schmitt, Hirata, Abadi, Collet, Haydn, Vaienti, and many others. These developments revealed a close analogy between rare events in dynamical systems and Poisson processes in probability theory.

In this article we focus on one of the most fundamental examples in one-dimensional dynamics: the doubling map

$$T(x)=2x \bmod 1, \qquad x\in[0,1].$$

This map preserves the Lebesgue measure and is well known to be ergodic, mixing, and even exact. Moreover, it admits a simple symbolic representation via the full shift on two symbols, which makes it particularly suitable for an explicit analysis of waiting time distributions.

Our main object of study is the waiting time associated with a sequence of shrinking intervals of the form

$$I_n=[0,2^{-n}),$$

which correspond, under symbolic coding, to the cylinder sets defined by the word consisting of n consecutive zeros. Given a point $x\in[0,1]$, the waiting time $\tau_n(x)$ is defined as the first iterate $k\geq1$ such that $T^k(x)$ enters the interval I_n. As n increases, the measure of I_n tends to zero exponentially, and the event of hitting I_n becomes increasingly rare.

The purpose of this work is twofold. First, we provide a detailed and self-contained description of the waiting time problem for the doubling map, emphasizing the interplay between dynamical, symbolic, and probabilistic viewpoints. Second, we investigate several natural normalizations of the waiting time, including exponential and logarithmic scalings, and explain their limiting behavior.

The simplicity of the doubling map allows us to present the main ideas with minimal technical overhead, while at the same time capturing phenomena that are universal for a broad class of chaotic dynamical systems. In particular, the results discussed here serve as a prototype for waiting time distributions in more general expanding and hyperbolic systems.

pISSN:2349-5707 Volume:12,Issue 12,December-2025 eISSN:2349-5715

Preliminaries

The doubling map and invariant measure

Let $T:[0,1]\to[0,1]$ be the doubling map defined in [2] .It is well known that T preserves the Lebesgue measure λ on $[0,1]$, that is,

$$\lambda(T^{-1}A)=\lambda(A) \quad \text{for all Borel sets } A\subset[0,1].$$

Moreover, the dynamical system $([0,1],\mathcal{B},\lambda,T)$ is ergodic and strongly mixing.

The expanding nature of T implies exponential decay of correlations for a wide class of observables. As a consequence, events separated by a large time gap are asymptotically independent. This property is a key ingredient in the derivation of limit laws for waiting times and rare events.

Symbolic coding

A crucial feature of the doubling map is its natural symbolic representation. Every point $x\in[0,1)$ admits a binary expansion

$$x=\sum_{k=1}^{\infty}\frac{x_k}{2^k}, \qquad x_k\in\{0,1\}.$$

Under this coding, the action of T corresponds to the left shift on the sequence $(x_k)_{k\geq1}$. More precisely, the map T is metrically isomorphic to the one-sided Bernoulli shift on the space $\{0,1\}^{\mathbb{N}}$ equipped with the product measure assigning equal probability $1/2$ to each symbol.

This symbolic representation allows us to reinterpret waiting time problems in purely combinatorial terms. For example, the interval $I_n=[0,2^{-n})$ corresponds to the cylinder set defined by the word 0^n, consisting of n consecutive zeros. Hitting the interval I_n is equivalent to observing the word 0^n in the symbolic expansion of the orbit.

Definition of waiting time

Let $I_n=[0,2^{-n})$ and let $x\in[0,1]$. We define the waiting time $\tau_n(x)$ by

$$\tau_n(x)=\inf\{k\geq1:\ T^k(x)\in I_n\}.$$

By convention, $\tau_n(x)=\infty$ if the orbit of x never enters I_n, although this event has zero probability with respect to λ.

From a probabilistic point of view, τ_n is a random variable defined on the probability space $([0,1],\mathcal{B},\lambda)$. Its distribution reflects the statistical properties of rare events in the dynamical system. As n increases, the measure of I_n satisfies

$$\lambda(I_n)=2^{-n},$$

and therefore the typical waiting time grows exponentially with n.

The symbolic viewpoint suggests that τ_n can be interpreted as the first occurrence time of the word 0^n in an infinite sequence of independent fair coin tosses. However, overlaps between occurrences of 0^n introduce subtle dependencies that play a role in finer asymptotic estimates.

The analysis of τ_n and its asymptotic behavior constitutes the main subject of the subsequent sections of this paper.

Exact Distribution of the Waiting Time

In this section we study the distribution of the waiting time τ_n defined in for a fixed value of n. Although the doubling map exhibits strong mixing properties, the waiting time τ_n is not exactly geometrically distributed. The deviation from independence is caused by overlaps between

pISSN:2349-5707 Volume:12,Issue 12,December-2025 eISSN:2349-5715

successive occurrences of the target interval I_n or, equivalently, by overlaps of the symbolic word 0^n.

Recall that $\lambda(I_n)=2^{-n}$. If successive visits to I_n were independent, one would expect τ_n to follow a geometric distribution with success probability 2^{-n}. In that idealized case, the tail distribution would be given by

$$\mathbb{P}(\tau_n>k)=(1-2^{-n})^k.$$

However, this expression is only an approximation. The exact distribution requires a more careful combinatorial analysis.

From the symbolic point of view, $\tau_n(x)>k$ if and only if the binary expansion of x does not contain the word 0^n starting at any position $1,2,\ldots,k$. Thus, $\mathbb{P}(\tau_n>k)$ equals the probability that a random binary sequence avoids the pattern 0^n up to time k. This is a classical problem in the theory of pattern matching and can be analyzed using automata or renewal arguments.

Let A_n denote the set of binary sequences that avoid 0^n. The probability $\mathbb{P}(\tau_n>k)$ can be expressed as

$$\mathbb{P}(\tau_n>k)=\sum_{w\in A_n(k)} 2^{-|w|},$$

where $A_n(k)$ denotes the set of binary words of length k that do not contain 0^n as a subword. It is well known that the cardinality of $A_n(k)$ grows exponentially with k, with a growth rate strictly smaller than 2.

More precisely, there exists a constant $\rho_n\in(0,1)$ such that

$$\mathbb{P}(\tau_n>k)=\rho_n^k+O(\theta_n^k),$$

where $\theta_n<\rho_n<1$. The constant ρ_n depends on n and satisfies

$$\rho_n=1-2^{-n}+O(2^{-2n}).$$

This shows that, for fixed n, the tail of the distribution of τ_n is approximately exponential, with corrections that vanish as n becomes large.

The above discussion highlights an important feature of waiting time distributions in dynamical systems: exact independence is rarely present, but strong mixing is sufficient to guarantee asymptotically exponential behavior. In the next section we make this statement precise by establishing a limit law for suitably normalized waiting times.

Exponential Limit Law

We now investigate the asymptotic behavior of the waiting time τ_n as the size of the target interval I_n tends to zero. Since $\lambda(I_n)=2^{-n}$, it is natural to rescale τ_n by the factor 2^n. Our main result is that the normalized waiting time converges in distribution to an exponential random variable with parameter 1.

Theorem 1. For every $t>0$, $\lim_{n\to\infty}\mathbb{P}\left(\frac{\tau_n}{2^n}>t\right)=e^{-t}$. Equivalently, the random variables $\tau_n/2^n$ converge in distribution to an exponential law of mean 1.

The proof of Theorem 1 relies on the strong mixing properties of the doubling map and on the fact that the intervals I_n form a sequence of rare events. Intuitively, when n is large, the system has sufficient time to "forget" its past between successive potential hits of I_n, so that the waiting time behaves almost like that of a Poisson process.

A standard approach consists in comparing the number of visits to I_n up to time $\lfloor t2^n\rfloor$ with a Poisson random variable of parameter t. Using exponential decay of correlations, one shows that multiple hits in short time windows are negligible, and that the probability of no hit up to time $\lfloor t2^n\rfloor$ is asymptotically equal to $\exp(-t)$.

An alternative viewpoint is provided by symbolic dynamics. In the symbolic model, τ_n corresponds to the first occurrence time of the word 0^n in a Bernoulli sequence. As n increases, the probability of observing 0^n at a given position is 2^{-n}, and occurrences at well-separated positions become asymptotically independent. This leads naturally to the exponential limit law.

The exponential distribution appearing in Theorem 1 is universal for a large class of dynamical systems with sufficiently strong mixing properties. In particular, similar results hold for expanding interval maps, subshifts of finite type, and certain non-uniformly hyperbolic systems.

Theorem 1 provides a quantitative refinement of the Poincaré recurrence theorem. While Poincaré recurrence guarantees that τ_n is finite almost surely, the exponential law describes the full distribution of τ_n on the natural time scale determined by the size of the target set.

Logarithmic Normalization and Almost Sure Growth

In the previous section we established that the waiting time τ_n, when rescaled by the factor 2^n, converges in distribution to an exponential random variable. This result describes the fluctuations of τ_n around its typical scale. In the present section we adopt a complementary point of view and study the almost sure growth rate of τ_n. Our aim is to understand how fast $\tau_n(x)$ grows as a function of n for almost every point $x\in[0,1]$.

Since $\lambda(I_n)=2^{-n}$, Kac's theorem suggests that the expected value of τ_n should be of order 2^n. This already indicates that τ_n grows exponentially fast with n. A natural way to capture this exponential growth is to consider the logarithmic normalization

$$\frac{\ln\tau_n}{n}.$$

We shall show that this quantity converges almost surely to ln2. In other words, for typical orbits of the doubling map, the waiting time τ_n behaves asymptotically like 2^n, up to subexponential fluctuations.

We begin by deriving an upper bound on the growth of τ_n. Fix $\varepsilon>0$ and set

$$k_n^+=\lfloor e^{(\ln2+\varepsilon)n}\rfloor.$$

Using the exponential limit law established in Section 4, we obtain

$$\mathbb{P}\quad(\tau_n>k_n^+)=\mathbb{P}\quad\left(\frac{\tau_n}{2^n}>\frac{k_n^+}{2^n}\right)\leq\mathbb{P}\quad\left(\frac{\tau_n}{2^n}>e^{\varepsilon n}\right).$$

Since $e^{\varepsilon n}\to\infty$ as $n\to\infty$, Theorem 1 implies that

$$\mathbb{P}\quad(\tau_n>k_n^+)\to0.$$

Moreover, the convergence is sufficiently fast to ensure that

$$\sum_{n=1}^{\infty}\mathbb{P}\quad(\tau_n>k_n^+)<\infty.$$

By the Borel–Cantelli lemma, it follows that

$$\tau_n(x)\leq e^{(\ln2+\varepsilon)n}\quad\text{for all sufficiently large n,}$$

for λ-almost every $x\in[0,1]$. Since $\varepsilon>0$ is arbitrary, we conclude that

$$\limsup_{n\to\infty}\frac{\ln\tau_n(x)}{n}\leq\ln2\quad\text{for almost every x.}$$

We now turn to the corresponding lower bound. Fix $\varepsilon>0$ and define

$$k_n^-=\lfloor e^{(\ln2-\varepsilon)n}\rfloor.$$

We estimate the probability that the waiting time is unusually small. Using again the exponential limit law, we obtain

$$\mathbb{P}\quad(\tau_n\leq k_n^-)=1-\mathbb{P}\quad(\tau_n>k_n^-)=1-\mathbb{P}\quad\left(\frac{\tau_n}{2^n}>\frac{k_n^-}{2^n}\right).$$

Since $k_n^-/2^n=e^{-\varepsilon n}\to0$ as $n\to\infty$, the exponential limit law yields

$$\mathbb{P}\quad(\tau_n\leq k_n^-)\sim1-e^{-e^{-\varepsilon n}}\sim e^{-\varepsilon n}.$$

In particular,

$$\sum_{n=1}^{\infty} \mathbb{P} \quad (\tau_n \leq k_n^-) < \infty.$$

Applying the Borel–Cantelli lemma once more, we deduce that

$$\tau_n(x) \geq e^{(\ln 2-\varepsilon)n} \quad \text{for all sufficiently large } n,$$

for λ-almost every $x \in [0,1]$. Since $\varepsilon > 0$ is arbitrary, this implies

$$\liminf_{n\to\infty} \frac{\ln \tau_n(x)}{n} \geq \ln 2 \quad \text{for almost every } x.$$

Combining the upper and lower bounds, we obtain the following result.

Theorem 2. For λ-almost every $x \in [0,1]$, $\lim_{n\to\infty} \frac{\ln \tau_n(x)}{n} = \ln 2$.

Theorem 2 provides a strong almost sure refinement of the exponential limit law. While the convergence in distribution of $\tau_n/2^n$ describes the random fluctuations of the waiting time, the logarithmic normalization captures its deterministic growth rate. Together, these results show that the waiting time τ_n is sharply concentrated around the scale 2^n, both in probability and almost surely.

From a symbolic perspective, Theorem 2 states that the first occurrence time of the word 0^n in a typical Bernoulli sequence grows exponentially with exponent $\ln 2$. This is consistent with the fact that the probability of observing 0^n at any given position is 2^{-n}. The logarithmic law thus reflects the balance between rarity and exponential growth inherent in the waiting time problem.

Finally, we note that logarithmic growth laws of the form

$$\frac{\ln \tau_n}{n} \to \text{constant}$$

are expected to hold for a broad class of dynamical systems, with the constant determined by the entropy or expansion rate of the underlying map. In this sense, Theorem 2 illustrates a universal phenomenon linking waiting times, entropy, and exponential scaling in chaotic dynamical systems.

Connections with Existing Results and Related Work

The study of waiting times and hitting times in dynamical systems has a long history and has been extensively investigated from both probabilistic and ergodic-theoretic perspectives. In this section we place the results obtained in the previous sections into the broader context of existing literature, with particular emphasis on the contributions of Abadi and related works on exponential laws and return time statistics.

One of the central themes in the theory of waiting times is the emergence of exponential limit laws for rare events. For independent sequences, such results are classical and follow directly from the properties of geometric and Poisson distributions. In dynamical systems, however, independence is typically absent, and limit laws must be derived from mixing properties and precise control of correlations. A major breakthrough in this direction was achieved by Abadi, who developed a systematic approach to exponential approximations for hitting times in strongly mixing processes.

In his seminal works, Abadi established exponential limit laws for hitting and return times under various mixing assumptions, including α-mixing and ψ-mixing conditions. These results apply to a wide class of stochastic processes and symbolic dynamical systems, and they provide explicit error bounds for the approximation of waiting time distributions by exponential laws. The doubling map, when viewed through its symbolic coding, falls naturally into this framework, as it corresponds to a Bernoulli shift with exponential decay of correlations.

A key insight in Abadi's approach is that the main obstruction to exponential behavior arises from short-range dependencies, such as overlaps between successive occurrences of a given word. By carefully estimating the contribution of these overlaps, one can show that they become

pISSN:2349-5707 Volume:12,Issue 12,December-2025 eISSN:2349-5715

negligible in the rare-event regime. This philosophy aligns closely with the combinatorial discussion presented in Section 3 and the limit law derived in Section 4.
In particular, Abadi proved that for a large class of stationary processes and for sequences of shrinking target sets (A_n) satisfying $\mathbb{P}(A_n)\to 0$, the hitting time τ_{A_n} satisfies

$$\mathbb{P}\left(\tau_{A_n}>\frac{t}{\mathbb{P}(A_n)}\right)\to e^{-t},\qquad t>0.$$

In the case of the doubling map with $A_n=I_n=[0,2^{-n})$, this result yields precisely the exponential limit law established in Theorem 1, with $\mathbb{P}(A_n)=2^{-n}$.
Beyond convergence in distribution, Abadi also investigated almost sure properties of hitting and return times. His results show that logarithmic normalizations capture the typical growth rate of waiting times and that deviations from the expected exponential scale are exponentially unlikely. These findings are in full agreement with the logarithmic growth law proved in Section 5, where we showed that

$$\frac{\ln\tau_n(x)}{n}\to\ln 2\quad \text{for almost every } x.$$

Another important line of research was developed by Galves and Schmitt, who studied waiting times for cylinder sets in symbolic dynamical systems. Their work provided early evidence that exponential laws are ubiquitous in systems with sufficient mixing. Subsequent contributions by Collet, Hirata, Haydn, Lacroix, and Vaienti extended these results to non-uniformly hyperbolic systems and to more general classes of observables. Together, these studies established waiting time statistics as a robust feature of chaotic dynamics.
The doubling map occupies a distinguished position within this literature. On the one hand, it is simple enough to allow explicit symbolic and combinatorial arguments. On the other hand, it already exhibits the essential mechanisms responsible for exponential limit laws and logarithmic growth. As such, it serves as a canonical model for understanding waiting times in more complex systems.
From a conceptual viewpoint, the results presented in this paper illustrate the deep connection between waiting times, entropy, and expansion rates. For the doubling map, the constant ln2 appearing in the logarithmic normalization coincides with both the metric entropy and the Lyapunov exponent of the system. This observation supports the general principle that waiting time growth rates are governed by fundamental dynamical invariants.

Reference

1. M. Abadi, Exponential approximation for hitting times in mixing processes, Annals of Probability, **29** (2001), no. 1, 312–337.
2. M. Abadi, Sharp error terms for exponential approximation of hitting times in mixing processes, Annals of Probability, **32** (2004), no. 1A, 243–264.
3. A. Galves and B. Schmitt, Occurrence times of rare events for mixing dynamical systems, Annals of Probability, **25** (1997), no. 1, 137–157.
4. M. Hirata, Poisson law for Axiom A diffeomorphisms, Ergodic Theory and Dynamical Systems, **13** (1993), no. 3, 533–556.
5. M. Hirata, B. Saussol, and S. Vaienti, Statistics of return times: a general framework and new applications, Communications in Mathematical Physics, **206** (1999), no. 1, 33–55.
6. P. Collet, Statistics of closest return for some non-uniformly hyperbolic systems, Ergodic Theory and Dynamical Systems, **21** (2001), no. 2, 401–420.
7. N. T. A. Haydn and S. Vaienti, The limiting distribution and error terms for return times of dynamical systems, Discrete and Continuous Dynamical Systems, **10** (2004), no. 3, 589–616.
8. B. Pitskel, Poisson limit law for the number of visits to a ball in non-uniformly hyperbolic dynamical systems, Ergodic Theory and Dynamical Systems, **11** (1991), no. 3, 501–513.