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FORMULATION OF THE CAUSHIE PROBLEM FOR THE TIME-FRACTAL AIRY EQUATION ON A BOUNDARY GRAPH

Kamoliddin Rakhimov Orinbayevich

Head of the Department of Exact Sciences, University of Exact and Social Sciences, PhD

Mashrapov Kuvonchbek Qakhramon ugli

Master's student, University of Exact and Social Sciences.

Abstract

The formulation of the Cauchy problem for the time-fractional Airy equation on a bounded graph and the proof of the uniqueness of the solution by the method of potentials through boundary conditions are considered.

Keywords:

Airy equation, ibvp, , fundamental solutions, Kirchhoff's rule at a point

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INTRODUCTION

In recent years, research at the intersection of differential equation theory and graph theory has been playing an important role in the development of science. In particular, due to the increasing need to model processes in objects with complex geometric structures, the study of differential equations given in graphs is becoming an urgent issue. Such problems are not only of theoretical importance, but are also widely used in the analysis of physical processes, engineering systems, transport and communication networks.

In this work, the formulation of the Cauchy problem for a fractional derivative differential equation given in a finite star graph is considered. A star graph is a structure in which several intersections (edges) are interconnected through a single node (central point). Although such a graph looks simple, the process of solving the differential equations given in it has its own complexities, since it is necessary to determine special connection (connection) conditions at the node point.

In this study, we consider a simple star-shaped graph formed by connecting two finite sections and one half-line at a single point, called the vertex of the graph. For the fractional differential equation given in this graph, the correct formulation of the Cauchy problem, that is, the existence, uniqueness, and continuous dependence of the solution, is of great importance.

Fractional differential equations, unlike classical differential equations, are distinguished by the fact that they take into account the "memory effect" of processes. Therefore, they are widely used in modeling anomalous diffusion, viscoelastic media, and complex dynamical systems. By studying such equations in star-shaped graphs, it is possible to gain a deeper understanding of the laws of propagation of processes in branched systems in time and space.

The main focus of the work is on the formulation of the Cauchy problem, determining the differential equation for each edge of the graph and giving the corresponding connection

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conditions at the nodes. In addition, the possibilities of using the potential method in solving the problem are also considered. This approach is an effective tool for constructing analytical solutions to the problem and analyzing their properties.

MAIN PART

The graph Γ has connections , Ω = Γ+ ∪ Γ- , k incoming and m outgoing edges. On the incoming edges, we denote the Γ- coordinates as L_j (0, L_j), j = 1, 2, and on the outgoing edges, we denote the Γ+ coordinates as 0 to L_j (L_j > 0, j = k + 1, k + m). We denote the edges of the graph by B_j, where j = 1, k + m ().(k=1,2 m=1)

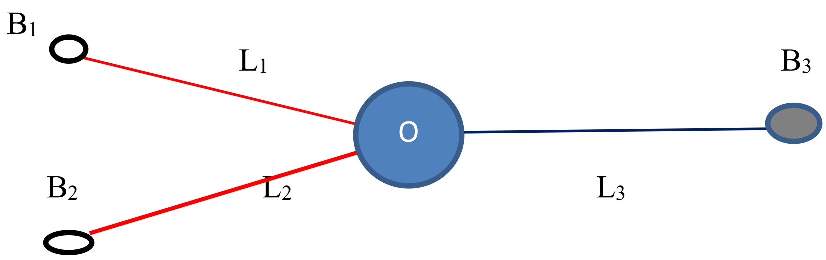

Figure 2.4.1

Let's assume, $f = (f_1, f_2, f_3)^T$), $u^- = (u1, u2)^T$, $u^+ = u_3$ va u = $\begin{matrix} u^- \\ u^+ \end{matrix}$

We consider the fractional derivative of the Airy equation with respect to time for each edge of the graph in the following form:

$$D_{0t}^{\alpha} u(x,t) - \frac{\partial^3}{\partial^3 x} u(x,t) = f_j(x,t) \quad 0 < t < T, x \quad B_j \qquad (2.4.1)$$

Initial conditions:

$$u(x,0) = u_0(x), x \quad \bar{B}_j, j = \overline{1,3} \quad (2.4.2)$$

here $u_0 = (u_0^1, u_0^2, u_0^3)^T$

Let the following conditions be satisfied at the intersection points of the graph::

$$Au(0,t) = 0, 0 < t < T \quad (2.4.3)$$

$$u_x^+(0,t) = Bu_x^-(0,t) \quad (2.4.4)$$

here $A = \begin{matrix} 0 & 0 & 0 \\ 1 & -a_2 & 0 \\ 1 & 0 & -a_3 \end{matrix} \quad B = (b_1, b_2)$

We require that the conditions of Kirchhoff's law be fulfilled at the intersection point of the ends:

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$$C^{-}=(\frac{1}{a_1},\frac{1}{a_2}),C^{+}=\frac{1}{a_3},a_1=1,a_j\quad 0, \qquad (2.4.5)$$

$$u_{j,0}(x)\quad C(\overline{b_j})f_j(x,t)\quad C^{0,1}(\overline{b_j}\quad [0.T])$$

$u(x,t)|_{\partial\Gamma}=\varphi(t),\frac{\partial}{\partial x}u^{-}(x,t)=\phi(t),0<t<T,$ (2.4.6)here And for j=3 $\varphi=(\varphi_1,\varphi_2,\varphi_3)^T$ and $\varphi^{-}=(\varphi_1,\varphi_2,)^T$.(2.4.3) at3 ta (k+m)The equation exists, but there are 2 (k+m-1) conditions.

Teorema 1. If BTB-Ik is negative, $u_{j,0}(x)\quad C(\overline{b_j})f_j(x,t)\quad C^{0,1}(\overline{b_j}\quad [0.T])$ functions and $\varphi(t)$ va $\phi(t)$ [0,T] is differentiable on the interval, it has a unique solution satisfying equalities 2.4.1-2.4.6.

Using the method of potentials, we found the fundamental solutions of equation 2.4.1 in the form A.Psxu:

$$G_{\alpha}^{2\alpha/3}(x,t)=\begin{matrix}\phi(-\alpha/3,2\alpha/3,x/t^{\alpha/3}) & x<0\\ -2\operatorname{Re}[e^{2\pi i/3}\phi(-\alpha/3,2\alpha/3,e^{2\pi i/3}x/t^{\alpha/3})] & x>0\end{matrix} \qquad (2.4.7)$$

Using the above equation, we can write the second fundamental solution as follows:

$$V_{\alpha}^{2\alpha/3}(x,t)=\frac{1}{3t^{t-\alpha/3}}\operatorname{Im}[e^{2\pi i/3}\phi(-\alpha/3,2\alpha/3,e^{2\pi i/3}x/t^{\alpha/3})] \quad (2.4.8)$$

Using the above fundamental solutions, we can find the general solutions of equation 2.4.1 in the formin:

$$u_j(x,t)=\int_0^t G_{\alpha}^{2\alpha/3}(x-L_j,t-\tau)\alpha_j(\tau)d\tau+\int_0^t V_{\alpha}^{2\alpha/3}(x-L_j,t-\tau)\beta_j(\tau)d\tau+\int_0^t G_{\alpha}^{2\alpha/3}(x-0,t-\tau)\gamma_j(\tau)d\tau+\int_0^t V_{\alpha}^{2\alpha/3}(x-0,t-\tau)\rho_j(\tau)d\tau+F_j(x,t)\quad j=\overline{1,k+}$$

and here a α_j,γ_j(j= $\overline{1,k+m}$),β_j(j= $\overline{1,k}$), ρ_j(j= $\overline{k,k+m}$)for unknown functions), ρ_j(t)=0), (j=$\overline{1,k}$), ;),β_j=0(j=$\overline{k=1,k+m}$), appropriate.

CONCLUSION

In this work, the formulation of the Cauchy problem for a fractional differential equation given on a finite star graph and its main properties were studied in detail. During the research, a mathematical model of the problem was precisely formulated, a system of equations was constructed for each edge of the graph, and the connection conditions at the central node were justified. This is one of the important stages necessary for the correct formulation of the problem.

As a result of the analysis, the existence and uniqueness of the solution to the problem under consideration were proven. That is, it was shown that, based on the given initial conditions, this differential equation has a unique solution. This result is one of the fundamental properties of the Cauchy problem, confirming its mathematically stable and correct formulation.

Also, in the process of work, it was found that the existence and uniqueness of the solution directly depend on certain conditions, in particular, on the coefficients in the equation and their properties. This relationship was substantiated by Theorem 1 and the conditions under which the solution remains stable were explained. The results show that when conditions such

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as continuity and boundedness of the coefficients are met, the solution to the problem not only exists, but is also unique.

In addition, the use of the potential method has proven to be an effective approach to solving the problem. This method allows for a constructive construction of the solution and a deeper analysis of its properties. This creates a basis for the development of new methodological approaches to the study of fractional differential equations given on graphs.

In general, the results of this study make a significant contribution to the study of differential equations on finite star graphs. The results obtained can be applied in more complex graph structures, under various boundary conditions, and in practical problems. At the same time, they serve as a theoretical basis for further scientific research in this area.

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