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INVERSE PROBLEM SOLUTION FOR THE HEAT EQUATION

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Abstract

This article examines the inverse problem for the heat conduction equation. The primary objective of the research is to develop a method for reconstructing an unknown boundary function based on additional temperature data from an internal point of the object. The Fourier method of separation of variables was applied to solve the problem, which subsequently reduced it to a Volterra integral equation of the second kind. The proposed solution algorithm was numerically implemented in the Python programming language using the trapezoidal rule, and the obtained results were analyzed.

Keywords

Heat equation, inverse problem, Fourier method, Volterra integral equation, numerical solution, Python, heat flux, mathematical modeling.

Introduction Today, mathematical modeling of processes in thermal physics and engineering has reached a new level. In particular, solving inverse problems for the heat equation is of great importance not only from a theoretical but also from a practical perspective.

Technological monitoring and diagnostics: In many complex systems (rocket engines, chemical reactors, nuclear facilities), it is impossible to measure the internal thermal state directly. In such conditions, reconstructing the entire process based on external boundary data or temperature values at a specific internal point is the only effective solution. This study proposes a solution method by reducing the problem to a Volterra integral equation. Accurate prediction of heat flows prevents material overheating and allows for the optimization of energy consumption. This is crucial for reducing production costs and increasing system safety. The proposed algorithm serves as a theoretical foundation and practical tool for solving complex inverse problems in fields such as thermal engineering, aerodynamics, and geophysics.

Problem Statement Find a function $u\ x\ t(\ ,\ )$ that satisfies the heat conduction equation in the domain $= \quad \{0\ x \quad ,0 \quad t\ T\}$

$u\ a_t = {}^2u_{xx}$ (1)

the initial condition

$u\ |_{t=0} = 0,$ (2)

and the boundary conditions

$u\ |_{x=0} = \quad (\ )t\ ,\ u|_{x=} \ = 0$ (3)

Inverse problem. Suppose g(t) is an unknown function in the problem (1)-(3). Given the additional condition on the solution of (1)-(3)

$$u\,|_{x\ x=0} = \ (\ )t\ ,0 \qquad x0 \qquad (4)$$

Find the function $(\ )t$ if an **additional condition** is given. In this case, it is assumed that the **smoothness condition** $(0) = 0$ is satisfied.

Reducing the boundary conditions to a homogeneous state

To solve the problem using the **Fourier method**, we must reduce the boundary conditions to zero. For this purpose, the auxiliary To solve the inverse problem, we first solve the direct problem (1)-(3). We apply the Fourier method to solve the problem; for this purpose, we must reduce the boundary conditions to a homogeneous state. Auxiliary

$$V\,x\,t(\ ,\ ) = u\,x\,t(\ ,\ ) + \ (\ )t\ \overset{x^{-}}{} \qquad (5) \qquad \overline{}$$

we introduce the function, resulting in $V(0,t) = 0$ and $V(\ ,t) = 0$. By calculating the necessary derivatives from equation (5), we arrive at the following problem for the function $V\,x\,t(\ ,\ )$ in the form of a **non-homogeneous heat equation**:

$$Vt = a^{2}\,Vxx + \frac{x-}{}\ (\ )t$$

$$V\,x(\ ,0) = 0,\ V(0,t) = 0,\ V(\ ,t) = 0 \qquad (7)$$

To find the solution $V\,x\,t(\ ,\ )$ for the problem (6)-(7), we expand it into a series using the eigenfunctions $X_n(\ )x = \sin(nx)$:

$$V\,x\,t(\ ,\ ) = \sum_{n=1} T\,t_n(\ )\sin(nx) \qquad (8)$$

We also expand the function on the right side of the equation, $f\,x\,t(\ ,\ ) = \frac{x-}{}\ (\ )t$,

into a Fourier series:

$$f\,x\,t(\ ,\ ) = \sum_{n=1} f\,t_n(\ )\sin(nx)\ ,$$

where $f\,t_n(\ ) = \frac{2}{}\int_0 f\,x\,t(\ ,\ )\sin(nx\,dx)$. Based on these, the solution takes the following form:

$$(7)\ V\,x\,t(\ ,\ ) = \sum_{n=1} \int_0^{t} f_n(\ )\ e^{-a\,n2\,2(t-\ )}d\ \ \sin(nx)$$

Now:

$$f\,t_n(\,) = 2 \;-0 \quad x^{-} \qquad (\,)t\,\sin(nx\,dx) = \;-(\,)^{t} \;\; {}^{2}\; 0 \;\; {}^{(x-\;\;)}\sin(nx\,dx)$$

Calculating the integral:

$$- \qquad - \qquad 0 \;\; {}^{x} \;\; {}^{-} \;\; \overline{\ }\sin(nx\;dx) = {}^{1}n\;{}^{n}\;{}^{1}n$$
$$((\;1)- - =^{n}\;1) \qquad 0n^{4},,\;nn_{=} = +\,2\;\;.2kk\;\;1,$$
$$(1- -(\;1)\;)-$$

Resulting in:

$$\_\_\_\_8 \quad '(\;)t\;,\;n$$
$$= +2k \quad 1, f_n(t) = \qquad n$$
$$0,\;\; n = 2\;.k$$

Substituting this into the solution form (9):

$$t$$

$$V\,x\,t(\,,\,) = \quad k{=}0 \quad 0 \quad \_\_\_\_(28 \quad k \quad (\,)+t\;1)\;ea2(2k+1)\;(2 \quad -t)d \quad \sin(2k$$
$$+1)x =$$

$$t$$

$$= \quad (2k+1)\;e$$
$$0\;k\;0$$

According to equation (5), the solution to the problem (1)-(3) is:

$$t$$

$$u\,x\,t(\,,\,) = \qquad = \quad (2k+1)\;e$$
$$0\;k\;0$$

To find the solution for the inverse problem (1)-(4), we satisfy the additional condition (4) $u\;x$ $t(\,{}_0,\,) = \;\;(\,)t$, and derive a simplified series for $n = +2\;1k$ (odd numbers):

$$(\quad)t \quad = \quad k \quad =0 \quad \_\_\_\_ \quad (2k8+1) \quad 0t \qquad (\quad)\cancel{e-a2}(2k+1) \quad (2$$
$$t- \quad)d \quad \sin((2k+1)x0)+ \qquad -x0 \quad (t)$$

By applying the integration by parts method and considering the condition $(0)\,{}_{=}\,0$, we arrive at a **Volterra integral equation of the second kind** with respect to the unknown function $(\,)t$:

t

$$C \quad (\ )t - \quad 0\, K\, t(\ , \qquad) \quad (\ )d = (\ )t$$

Where:

- **Constant coefficient C:**

$$C = \quad - x0 + 8 \qquad = \sin((22kk\cancel{++11)}x0)$$

$$k\, 0$$

- **Kernel of the integral equation $K\, t(\ ,\ )$:** $8a2(2k$

$$K\, t(\ ,\quad) = \ \cancel{k \quad =0 \quad +1})e{-}a2(2k{+}1)\ (2\ t{-} \quad)\ \sin((2k+1)x0)$$

We calculate the approximate solution of this integral equation using quadrature formulas. For instance, we develop a program in the Python programming language using the trapezoidal rule to produce numerical and graphical results. The results obtained through this program are presented below.

Numerical Results

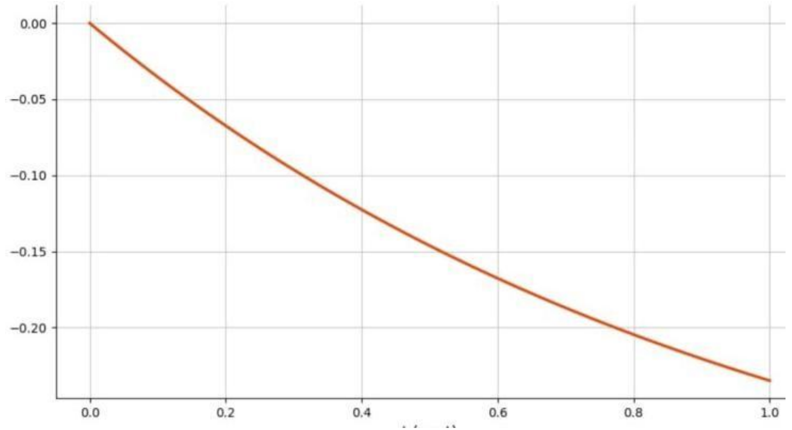
In this case, $(\ )t = -e^{-t} \quad 1$.

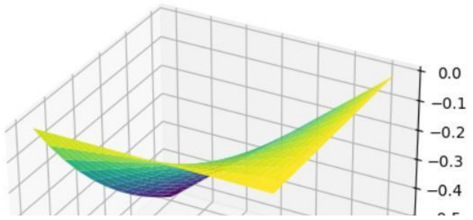
1. Inverse Problem (1)-(4) with $x_0 = =1, a \quad 1$.

Below are the values and the graph of the function $(\ )t$ within the interval $[0,T]$:

$(\ )t$ function values	$(\ )t$ function graph

t	$(\ )t$
0.1	-0.034696
0.2	-0.066091
0.3	-0.094498
0.4	-0.120201
0.5	-0.143459
0.6	-0.164503
0.7	-0.183545
0.8	-0.200775
0.9	-0.216365
1.0	-0.230471

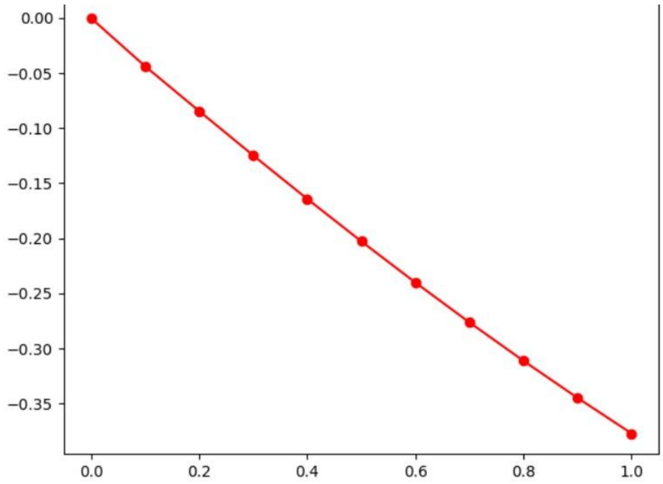


t	$(\ )t$	$u\ x\ t(\ ,\ )$	$u\ x\ t(\ ,\ )$ function graph
0.1	-0.034696	-0.046	
0.2	-0.066091	-0.0876	
0.3	-0.094498	-0.1254	
0.4	-0.120201	-0.1597	
0.5	-0.143459	-0.1907	

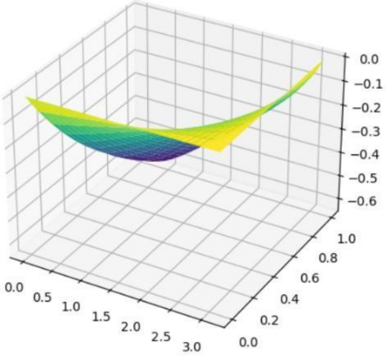
0.6	-0.164503	-0.2189	
0.7	-0.183545	-0.2445	
0.8	-0.200775	-0.2677	
0.9	-0.216365	-0.2888	
1.0	-0.230471	-0.3079	

2. Inverse Problem (1)-(4) with $x_0 =$ / 6,a =1/ 2 .

Below are the values and the graph of the function ( )t within the interval [0,T]:

<table>
<tr><th>( )<i>t</i> function values</th><th>( )<i>t</i> function graph</th></tr>
<tr><td>
<table>
<tr><td>t</td><td>( )<i>t</i></td></tr>
<tr><td>0.1</td><td>-0.0437</td></tr>
<tr><td>0.2</td><td>-0.0844</td></tr>
<tr><td>0.3</td><td>-0.1247</td></tr>
<tr><td>0.4</td><td>-0.1642</td></tr>
<tr><td>0.5</td><td>-0.2027</td></tr>
<tr><td>0.6</td><td>-0.2401</td></tr>
<tr><td>0.7</td><td>-0.2763</td></tr>
<tr><td>0.8</td><td>-0.3112</td></tr>
<tr><td>0.9</td><td>-0.3449</td></tr>
<tr><td>1.0</td><td>-0.3772</td></tr>
</table>
</td><td>


</td></tr>
</table>

t	( )t	$u\ x\ t$(,)	$u\ x\ t$(,) function graph
0.1	-0.0437	-0.047	
0.2	-0.0844	-0.0918	
0.3	-0.1247	-0.1347	
0.4	-0.1642	-0.1759	
0.5	-0.2027	-0.2559	
0.6	-0.2401	-0.2548	
0.7	-0.2763	-0.2929	
0.8	-0.3112	-0.3304	
0.9	-0.3449	-0.3673	
1.0	-0.3772	-0.4039	

Conclusion: The method of reducing the inverse problem for the heat conduction equation to a Volterra integral equation using Fourier series has shown its effectiveness. This approach significantly simplifies the solution of complex differential equations. The trapezoidal rule provided sufficient accuracy in the numerical solution of the integral equation constructed to determine the unknown boundary function. The software tool developed in Python enabled accurate prediction of temperature changes over various time intervals. The obtained results serve as an important practical tool for remote monitoring of technological processes (e.g., rocket engines or nuclear reactors), i.e., for reconstructing the thermal state at points where direct measurement is impossible. The numerical results and graphical illustrations show that the function values found based on the given additional conditions correspond to real physical processes, and this algorithm can be applied to solve complex problems in aerodynamics and geophysics.

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